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Absorption of a randomly accelerated particle: gambler's ruin in a different game

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Abstract. We consider a particle which is randomly accelerated by Gaussian white noise on the line $0 < x < 1$, with absorbing boundaries at $x = 0, 1$. Denoting the initial position and velocity of the particle by x_0 and v_0 and solving a Fokker–Planck-type equation, we derive the exact probabilities $q_0(x_0, v_0)$, $q_1(x_0, v_0)$ of absorption at $x = 0, 1$, respectively. The results are in excellent agreement with computer simulations.

A well known topic in random walk theory [1] is the problem of the 'gambler's ruin'. Initially, the gambler has an amount of money x_0 and the bank the amount $1-x_0$. The gambler flips a coin repeatedly, randomly winning or losing the increment ϵ. The game ends when the gambler's funds reach 0 or 1. The problem is to compute the probability $q_0(x_0)$ that the gambler loses everything.

The problem is easily solved. Since $q_0(x_0) = \frac{1}{2}[q_0(x_0+\epsilon) + q_0(x_0-\epsilon)]$,

$$\frac{\mathrm{d}^2 q_0(x_0)}{\mathrm{d}x_0^2} = 0 \tag{1}$$

in the limit $\epsilon \to 0$. From equation (1) and the boundary conditions $q_0(0) = 1$, $q_0(1) = 0$,

$$q_0(x_0) = 1 - x_0. \tag{2}$$

As the starting capital increases from 0 to 1, the probability of the gambler's ruin decreases from 1 to 0.

Instead of the gambling scenario one could equally well imagine a particle making a random walk with infinitesimal steps $\pm\epsilon$ on the x-axis, with initial position $0 < x_0 < 1$. Over the course of time the particle eventually arrives at $x = 0$ or 1. The quantities $q_0(x_0)$ in equation (2) and $q_1(x_0) = 1 - q_0(x_0)$ represent the probabilities that the particle first reaches the edge of the interval at $x = 0$ and 1, respectively. Alternatively, we could impose absorbing boundary conditions and interpret $q_0(x_0)$ and $q_1(x_0)$ as the probabilities of absorption at $x = 0$ and at $x = 1$.

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In this paper we also consider a particle on the finite interval $0 < x < 1$, but we assume that the changes in the *velocity* rather than the position of the particle are random. The particle moves according to the Langevin equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \eta(t) \tag{3}$$

where the acceleration $\eta(t)$ has the form of Gaussian white noise, with

$$\langle \eta(t) \rangle = 0 \qquad \langle \eta(t_1)\eta(t_2) \rangle = 2\delta(t_1 - t_2). \tag{4}$$

Imposing absorbing boundary conditions, we derive the probabilities $q_0(x_0, v_0)$, $q_1(x_0, v_0)$ of absorption at $x = 0$ and at $x = 1$, respectively, as functions of the initial position and velocity.

The quantity $q_0(x_0, v_0)$ can also be interpreted as the probability of a gambler's ruin, but the game is different. The gambler has an amount of money $x(t)$ at time t and the bank the amount $1 - x(t)$. Money is transferred from the bank to the gambler at a rate $v = \mathrm{d}x/\mathrm{d}t$, which may be positive or negative. At regular infinitesimal intervals the gambler flips a coin, randomly increasing or decreasing the rate v by the increment Δ. The game ends when x reaches 0 or 1. The quantity $q_0(x_0, v_0)$ is the probability that a gambler with initial conditions x_0, v_0 loses everything.

In the case of a random walk on the x-axis, the probability density $P(x, x_0, t)$ at time t of a particle which is initially at x_0 obeys the diffusion equation. For a particle which is randomly accelerated according to equations (3) and (4), the probability density $P(x, v; x_0, v_0; t)$ in the phase space (x, v) satisfies the Fokker–Planck equation [2]

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{\partial^2}{\partial v^2} \right) P(x, v; x_0, v_0; t) = 0 \tag{5}$$

corresponding to diffusion of the velocity, with the initial condition

$$P(x, v; x_0, v_0; 0) = \delta(x - x_0)\delta(v - v_0). \tag{6}$$

In analogy with the discussion leading to differential equation (1) for $q_0(x_0)$, let us consider a discrete dynamics in which the velocity v changes by $\pm\Delta$ with equal probability at time intervals τ. For this dynamics

$$P(x, v; x_0, v_0; t) = \tfrac{1}{2}[P(x - v\tau, v + \Delta; x_0, v_0; t - \tau) + P(x - v\tau, v - \Delta; x_0, v_0; t - \tau)] \tag{7}$$

$$q_0(x_0, v_0) = \tfrac{1}{2}[q_0(x_0 + v_0\tau, v_0 + \Delta) + q_0(x_0 + v_0\tau, v_0 - \Delta)]. \tag{8}$$

Expanding equations (7) and (8) in τ and Δ, dividing by τ, and taking the limit $\tau = \frac{1}{2}\Delta^2 \to 0$ gives us a 'poor man's' derivation of the Fokker–Planck equation (5) and the corresponding differential equation

$$\left(v_0 \frac{\partial}{\partial x_0} + \frac{\partial^2}{\partial v_0^2} \right) q_0(x_0, v_0) = 0 \tag{9}$$

for the probability of absorption at $x = 0$.

To solve equation (9) with the absorbing boundary condition

$$q_0(0, v_0) = 1 \qquad v_0 < 0 \tag{10}$$

and the requirements

$$q_0(x_0, v_0) = q_1(1 - x_0, -v_0) \tag{11}$$

$$q_0(x_0, v_0) + q_1(x_0, v_0) = 1 \tag{12}$$

of reflection symmetry and total probability equal to 1, we first make the substitution

$$\psi(x, v) = q_0(x, -v) - \tfrac{1}{2}. \tag{13}$$

Expressed in terms of $\psi(x, v)$, equations (9)–(12) take the form

$$\left(v\frac{\partial}{\partial x} - \frac{\partial^2}{\partial v^2}\right)\psi(x, v) = 0 \tag{14}$$

$$\psi(0, v) = \tfrac{1}{2} \qquad v > 0 \tag{15}$$

$$\psi(x, v) = -\psi(1 - x, -v). \tag{16}$$

Masoliver and Porrà [3] have shown how certain Fokker–Planck-type equations on the finite interval $0 < x < 1$ can be solved exactly. They derived an exact result for the average time $T(x_0, v_0)$ a randomly accelerated particle with initial conditions x_0, v_0 takes to reach a boundary of the interval. The probability that the particle has not yet reached a boundary after a time t decays as e^{-Et}, as discussed by Burkhardt [4]. He obtained E numerically with an approach similar to [3] and related it to the confinement free energy of a semiflexible polymer in a tube. In another application inspired by [3], Burkhardt *et al* [5] calculated the equilibrium distribution function $P(x, v)$ of a randomly accelerated particle on the line $0 < x < 1$ undergoing inelastic collisions at the boundaries [6].

The function $\psi(x, v)$ satisfies the same steady-state Fokker–Planck equation (14) as the quantity $P(x, v)$ considered in [5] and has the same Green's function solution

$$\begin{aligned}\psi(x, v) = {}& \frac{v^{1/2}}{3x}\int_0^\infty \mathrm{d}u\, u^{3/2}\mathrm{e}^{-(v^3+u^3)/9x}\, I_{-1/3}\left(\frac{2v^{3/2}u^{3/2}}{9x}\right)\psi(0, u) \\ & - \frac{1}{3^{1/3}\Gamma(\frac{2}{3})}\int_0^x \mathrm{d}y\, \frac{\mathrm{e}^{-v^3/9(x-y)}}{(x-y)^{2/3}}\, \frac{\partial\psi(y, 0)}{\partial v} \qquad v > 0\end{aligned} \tag{17}$$

derived in [5]. Equation (17) only holds for positive v. For negative v, $\psi(x, v)$ can be obtained from equation (17) using the antisymmetry (16) under reflection.

Equation (17) determines $\psi(x, v)$ for all $x > 0$ and $v > 0$ from $\psi(0, v)$ and $\partial\psi(x, 0)/\partial v$. The first of these functions is given in equation (15). To determine the second, we set $v = 0$ in equation (17), which yields

$$\psi(x, 0) = \frac{1}{3^{1/3}\Gamma(\frac{2}{3})}\left[x^{-2/3}\int_0^\infty \mathrm{d}u\, u\mathrm{e}^{-u^3/9x}\psi(0, u) - \int_0^x \frac{\mathrm{d}y}{(x-y)^{2/3}}\frac{\partial\psi(y, 0)}{\partial v}\right]. \tag{18}$$

Then, substituting equation (18) in the relation $\psi(x, 0) + \psi(1 - x, 0) = 0$, which follows from (16), and using $\partial\psi(y, 0)/\partial v = \partial\psi(1 - y, 0)/\partial v$, also a consequence of (16), we obtain

$$\int_0^1 \frac{\mathrm{d}y}{|x-y|^{2/3}}\frac{\partial\psi(y, 0)}{\partial v} = \int_0^\infty \mathrm{d}u\, u\left[\frac{\mathrm{e}^{-u^3/9x}}{x^{2/3}} + \frac{\mathrm{e}^{-u^3/9(1-x)}}{(1-x)^{2/3}}\right]\psi(0, u). \tag{19}$$

The solution to integral equation (19), derived, following [7], in appendix B of [5], is given by

$$\frac{\partial\psi(x, 0)}{\partial v} = \int_0^\infty \mathrm{d}u\, u\,[R(x, u) + R(1 - x, u)]\,\psi(0, u) \tag{20}$$

where

$$R(x, u) = \frac{1}{3^{5/6}\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}\frac{u^{1/2}\mathrm{e}^{-u^3/9x}}{x^{7/6}(1-x)^{1/6}}\,{}_1F_1\left(-\frac{1}{6}, \frac{5}{6}, \frac{u^3(1-x)}{9x}\right) \tag{21}$$

and ${}_1F_1(a; b; z)$ is the confluent hypergeometric function [8, 9].

Equations (17), (20), and (21) determine $\psi(x, v)$ for all x and v from $\psi(0, v)$ for $v > 0$, which is known from the absorbing boundary condition (15). Substituting equations (15) and (21) in (20) leads to

$$\frac{\partial\psi(x,0)}{\partial v} = \frac{1}{3^{1/6}\Gamma(\frac{1}{3})}[x(1-x)]^{-1/6} \tag{22}$$

and from (15), (17) and (22)

$$\psi(x,v) = \frac{1}{2} - \frac{1}{2\pi}\int_0^x \mathrm{d}y\, \frac{\mathrm{e}^{-v^3/9(y-x)}}{(y-x)^{2/3}}\,[y(1-y)]^{-1/6}. \tag{23}$$

Rewriting equation (23) in terms of $q_0(x_0, v_0)$ using (11)–(13), we obtain our main result

$$q_0(x_0,v_0) = 1 - q_0(1-x_0,-v_0) = \frac{1}{2\pi}\int_{x_0}^1 \mathrm{d}y\, \frac{\mathrm{e}^{-v_0^3/9(y-x_0)}}{(y-x_0)^{2/3}}\,[y(1-y)]^{-1/6} \qquad v_0 > 0 \tag{24}$$

analogous to the solution (2) of the traditional gambler's ruin problem.

For $x_0 = 1$ equation (24) reproduces the expected result $q_0(1, v_0) = 1 - q_0(0, -v_0) = 0$, $v_0 > 0$, corresponding to the immediate absorption of a particle that is initially at either boundary with velocity directed outward from the interval $0 < x < 1$. For $x_0 = 0$ and $v_0 = 0$ the integral in equation (24) can be evaluated, yielding

$$q_0(0,v_0) = 1 - q_0(1,-v_0) = 1 - \frac{2\times 3^{2/3}}{\Gamma(\frac{1}{6})}\,v_0^{1/2}\,{}_1F_1(\tfrac{1}{6};\tfrac{7}{6};-\tfrac{1}{9}v_0^3) \qquad v_0 > 0 \tag{25}$$

$$q_0(x_0,0) = 1 - q_0(1-x,0) = 1 - \frac{6\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{6})^2}\,x_0^{1/6}\,{}_2F_1(\tfrac{1}{6},\tfrac{5}{6};\tfrac{7}{6};x_0). \tag{26}$$

Here ${}_1F_1(a; b; z)$ and ${}_2F_1(a, b; c; z)$ are the confluent and ordinary hypergeometric functions [8, 9].

The probability $q_0(x_0, v_0)$ of absorption at the origin, obtained from equation (24) by numerical integration, is shown in figure 1. The probability decreases monotonically as x_0 increases with fixed v_0 and as v_0 increases at fixed x_0, as expected. The quantity $q_0(x_0, v_0)$ is a non-singular function of (x_0, v_0) except at the two boundary points $(0, 0)$ and $(1, 0)$. The curves for $x_0 = 0.0, 0.1, 0.3, 0.5$ become smoother near $v_0 = 0$ as x_0 increases, and for $x_0 = 0.5$, $q_0(x_0, v_0) - \frac{1}{2}$ is an odd function of v_0, as implied by equations (11) and (12).

The points in figure 1 show the results of computer simulations, which clearly are in excellent agreement with the analytical results. Our simulation routine is based on the exact solution [10]

$$P_{\mathrm{free}}(x,v;x_0,v_0;t) = \frac{\sqrt{3}}{2\pi t^2}\exp\left\{-\frac{3}{t^3}\left[(x-x_0-v_0t)(x-x_0-vt)+\frac{1}{3}(v-v_0)^2t^2\right]\right\} \tag{27}$$

of the Fokker–Planck equation (5) with initial condition (6) in the absence of boundaries. Trajectories with the probability distribution $P_{\mathrm{free}}(x_{n+1}, v_{n+1}; x_n, v_n; \Delta_{n+1})$ given by (27) are generated using the algorithm

$$x_{n+1} = x_n + v_n\Delta_{n+1} + \left(\frac{\Delta_{n+1}^3}{6}\right)^{1/2}\left(s_{n+1}+\sqrt{3}\,r_{n+1}\right) \tag{28}$$

$$v_{n+1} = v_n + (2\Delta_{n+1})^{1/2}\,r_{n+1} \tag{29}$$

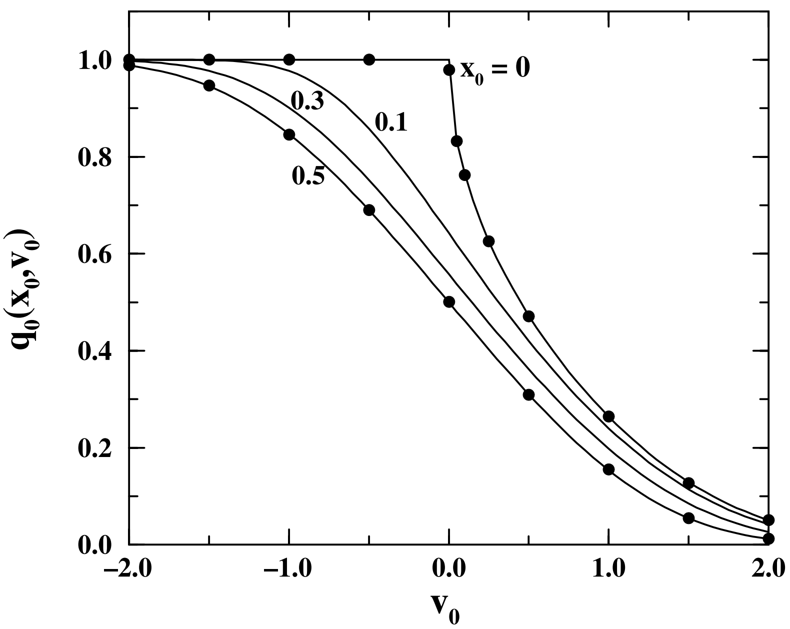

Figure 1. The probability $q_0(x_0, v_0)$ of absorption at the origin. The full curves show the exact result given in equation (24). The points are the results of our computer simulations. The data points have a statistical uncertainty $\pm\delta q_0$ with $|\delta q_0| \lesssim 0.001$.

where x_n and v_n are the position and velocity of the particle at time t_n, and $\Delta_{n+1} = t_{n+1} - t_n$. The quantities r_n and s_n are independent Gaussian random numbers such that

$$\langle r_n \rangle = \langle s_n \rangle = 0 \qquad \langle r_n^2 \rangle = \langle s_n^2 \rangle = 1. \tag{30}$$

In the absence of boundaries there is no time-step error in the algorithm, i.e. the Δ_n may be chosen arbitrarily. Close to boundaries small time steps are needed.

To derive a quantitative criterion for an acceptable time step, we begin with the averages

$$\langle x(t) \rangle = x_0 + v_0 t \qquad \langle [x(t) - \langle x(t) \rangle]^2 \rangle = \tfrac{2}{3} t^3 \tag{31}$$

implied by the distribution function (27). At time t the particle coordinate x has a Gaussian distribution, with a maximum at $x = x_0 + v_0 t$ and the root-mean-square width $\left(\frac{2}{3}t^3\right)^{1/2}$. The effect of the boundaries on the propagation is negligible if the Gaussian peak lies almost entirely within the interval $0 < x < 1$. This is certainly the case if, say,

$$0 < x_0 + v_0 t \pm 5t^{3/2} < 1. \tag{32}$$

Over the range of velocities encountered in our simulations, any t which satisfies the simpler, more stringent condition

$$t < \tfrac{1}{10} x_0 (1 - x_0) \tag{33}$$

also satisfies (32).

Keeping inequality (33) in mind, we performed our simulations with the time step

$$\Delta_{n+1} = 10^{-5} + 10^{-1} x_n (1 - x_n). \tag{34}$$

The time step decreases as the particle approaches the boundary and has the minimum value 10^{-5}. It is necessary to have a small non-zero minimum value. Otherwise the particle never arrives at the boundaries. Our results for the absorption probability $q_0(x_0, v_0)$ are averages based on 10^5 trajectories for each set of initial conditions x_0, v_0.

Finally, we note that $q_0(x_0, v_0)$ may be derived from another general Green's function solution† of the Fokker–Planck equation (14),

$$\psi(x, v) = \frac{v^{1/2}}{3x}\int_0^\infty \mathrm{d}u\, u^{3/2}\mathrm{e}^{-(v^3+u^3)/9x}\, I_{1/3}\left(\frac{2v^{3/2}u^{3/2}}{9x}\right)\psi(0, u)$$
$$+\frac{v}{3^{2/3}\Gamma(\frac{1}{3})}\int_0^x \mathrm{d}y\,\frac{\mathrm{e}^{-v^3/9(x-y)}}{(x-y)^{4/3}}\,\psi(y, 0) \qquad v > 0 \tag{35}$$

different from (17). By substituting equation (35) in (14), one can check that the Fokker–Planck equation is indeed satisfied. On the lines $x = 0$ and $v = 0$ equation (35) reduces to the identities $\psi(0, v) = \psi(0, v)$ and $\psi(x, 0) = \psi(x, 0)$, respectively.

Equation (35) determines $\psi(x, v)$ for all $x > 0$ and $v > 0$ from $\psi(0, v)$, $v > 0$ and $\psi(x, 0)$. The first of these functions is given in equation (15). To determine the second one, we differentiate equation (35) with respect to v and then set $v = 0$, which yields

$$\frac{\partial\psi(x, 0)}{\partial v} = \frac{1}{3^{2/3}\Gamma(\frac{1}{3})}\left[x^{-4/3}\int_0^\infty \mathrm{d}u\, u^2\mathrm{e}^{-u^3/9x}\psi(0, u) - 3x^{-1/3}\psi(0, 0)\right.$$
$$\left.-3\int_0^x \frac{\mathrm{d}y}{(x-y)^{1/3}}\,\frac{\partial\psi(y, 0)}{\partial y}\right]. \tag{36}$$

For the absorbing boundary condition (15) the first two terms on the right-hand side of (36) cancel. Substituting equation (36) in the relation $\partial\psi(x, 0)/\partial v - \partial\psi(1-x, 0)/\partial v = 0$, which follows from (16), using the invariance of $\partial\psi(y, 0)/\partial y$ under $y \to 1 - y$, and integrating with respect to x yields

$$\int_0^1 \mathrm{d}y\, |x-y|^{2/3}\,\frac{\partial\psi(y, 0)}{\partial y} = \text{constant}. \tag{37}$$

The function $\psi(x, 0)$ given in equations (13) and (26) satisfies equation (37). Substituting this $\psi(x, 0)$ and $\psi(0, v) = \frac{1}{2}$ into equation (35), integrating, and using (11)–(13), we obtain

$$q_0(x_0, v_0) = 1 - q_0(1 - x_0, -v_0)$$
$$= \frac{2\times 3^{1/3}v}{\Gamma(\frac{1}{6})^2}\int_{x_0}^1 \mathrm{d}y\,\frac{\mathrm{e}^{-v_0^3/9(y-x_0)}}{(y-x_0)^{4/3}}\,(1-y)^{1/6}\,{}_2F_1(\tfrac{1}{6}, \tfrac{5}{6}; \tfrac{7}{6}; 1-y) \qquad v_0 > 0. \tag{38}$$

With the help of the identity

$$\int_0^x \mathrm{d}y\,\frac{\mathrm{e}^{-v^3/9(x-y)}}{(x-y)^{2/3}}\,f(y) = \frac{v}{3^{2/3}\Gamma(\frac{1}{3})}\int_0^x \mathrm{d}z\,\frac{\mathrm{e}^{-v^3/9(x-z)}}{(x-z)^{4/3}}\int_0^z \mathrm{d}y\,\frac{f(y)}{(z-y)^{2/3}} \tag{39}$$

for arbitrary $f(y)$, one can convert expression (38) for $q_0(x_0, v_0)$ into the simpler form (24).

The second of the two Green function solutions (17), (35) looks simpler than the first, since no derivatives of ψ appear on the right-hand side, but our main result (24) for $q_0(x_0, v_0)$ is obtained more easily from (17).

† This solution may be derived by slightly modifying the derivation in appendix A of [5]. Setting $v = 0$ in equation (A3) of [5], solving for $W(s)$, and reinserting the result in (A3) with $v \neq 0$ yields the Laplace transform of the new solution (35).

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